千葉商大紀要　第 57 巻　第 2 号（2019 年 11 月）pp. 17-21

〔論　説〕

Right Inverse for Shift Operators

Take-Yuki NAGAO

1. Introduction

In this paper, we consider a shift operator S acting on an infinite dimensional linear space, and describe the solutions of the equation $(S-z)u=f$. Let us first be clear about what we mean by a shift operator. Let Ω be either $\mathbb{N}$ or $\mathbb{Z}$, and $\mathbb{K}$ be an algebraically closed field. We define $\mathbb{L}(\Omega,\mathbb{K})$ to be the linear space of everywhere defined mappings from Ω into $\mathbb{K}$. $\mathbb{L}(\Omega,\mathbb{K})$ is the linear space of lists or sequences of the form $(f_j)_{j\in\Omega}$ with values in $\mathbb{K}$.

In the former work, the author introduced the notion of hyperlists [1], which extends the notion of lists in computer science. $\mathbb{L}(\Omega,\mathbb{K})$ is a special case of a hyperlist. A shift is a mapping $\{f_j\}_j \mapsto \{f_{j-k}\}_j$ that is a commonly used operation in programming languages. Our purpose is to understand the behavior of the shift, when acting on an infinite list.

Given an integer $k\in\mathbb{Z}$, we consider the shift operator $S_k : \mathbb{L}(\Omega,\mathbb{K}) \longrightarrow \mathbb{L}(\Omega,\mathbb{K})$ defined by

$$(S_k f)_j = f_{j-k}, \quad j\in\Omega, \tag{1.1}$$

for any $f\in\mathbb{L}(\Omega,\mathbb{K})$. By convention, we set $f_\ell = 0$ if ℓ is undefined in Ω. Clearly, we have

$$S_k = (S_1)^k, \quad S_{-k} = (S_{-1})^k, \quad k\in\mathbb{N}. \tag{1.2}$$

The shift operator S_k on $\mathbb{L}(\Omega,\mathbb{K})$ can be classified into four classes:

(1) the identity operator ($k=0$ and $\Omega=\mathbb{N},\mathbb{Z}$),
(2) the right shift operator ($k>0$ and $\Omega=\mathbb{N}$),
(3) the left shift operator ($k<0$ and $\Omega=\mathbb{N}$),
(4) and the rotation operator ($k\neq 0$ and $\Omega=\mathbb{Z}$).

The purpose of this paper is to propose a unified method to describe the solutions of the equation $(S_k - z)u = f$ for all these classes.

The shift operator S_k is one of the simplest non-trivial operators acting on the lists, but not much is known about its spectral properties in the infinite dimensional case. It is shown below that a right inverse can be constructed, which is expected to help analyzing properties of shift operators.

2. Shift Operators

We define the operator P_k by $P_k = S_k S_{-k}$. The following lemma shows that P_k is a projection in $\mathbb{L}(\Omega,\mathbb{K})$.

Lemma 2.1. *The following statements hold.*

Date: 20 Sep., 2019.

(1) If $\Omega = \mathbb{N}$, then for any $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, we have

$$(P_k f)_j = \begin{cases} 0 & j < k \\ f_j & \text{otherwise.} \end{cases}$$

(2) If $\Omega = \mathbb{Z}$, then $P_k = 1$ for any $k \in \mathbb{Z}$.

Moreover, we have $P_k^2 = P_k$ for any $k \in \mathbb{Z}$.

Remark that by Lemma 2.1 we have

$$S_m S_{-m} = P_m, \quad S_{-m} S_m = 1, \quad m \in \mathbb{N},$$

and hence S_m is invertible if $\Omega = \mathbb{Z}$ but not invertible if $\Omega = \mathbb{N}$ and $m \neq 0$.

Theorem 2.2. *Assume either $\Omega = \mathbb{N}$ or $\Omega = \mathbb{Z}$. Let $k \in \mathbb{Z}$, $z \in \mathbb{K}$, and $u, f \in \mathbb{L}(\Omega, \mathbb{K})$. Consider the equation*

$$(S_k - z)u = f. \tag{2.1}$$

(1) If $k = 0$ and $z = 1$, then (2.1) is solvable if and only if $f = 0$. Any u is a solution if $f = 0$.

(2) If $k = 0$ and $z \neq 1$, then for any f, (2.1) has a unique solution u and $u = (1 - z)^{-1} f$.

(3) If $k > 0$ and $z = 0$, then (2.1) is solvable if and only if $f \in \operatorname{Ran} S_k$ and the solution is given by $u = S_{-k} f$. The solution is unique if exists.

(4) If $k < 0$ and $z = 0$, then for any f, (2.1) has a solution u that satisfies $u = S_{-k} f + g$, where $g \in \operatorname{Ker} S_k$ is arbitrary.

(5) If $k \neq 0$ and $z \neq 0$, then for any f, (2.1) has a solution u. Explicit formulae for the solutions are shown below:

(a) If $\Omega = \mathbb{N}$ and $k > 0$, then the solution u satisfies

$$u_{mk+\ell} = \frac{1}{z^m} \left(u_\ell - \sum_{j \in \mathbb{N}:\ 1 \leq j \leq m} z^{j-1} f_{jk+\ell} \right), \quad m, \ell \in \mathbb{N},\ 0 \leq \ell < |k|, \tag{2.2}$$

and

$$u_\ell = -\frac{f_\ell}{z}, \quad \ell = 0, 1, 2, \ldots, |k| - 1. \tag{2.3}$$

The solution is unique in this case.

(b) If $\Omega = \mathbb{N}$ and $k < 0$, then the solution u satisfies

$$u_{-mk+\ell} = z^m \left(u_\ell + \sum_{j \in \mathbb{N}:\ 0 \leq j < m} \frac{f_{-jk+\ell}}{z^{j+1}} \right), \quad m, \ell \in \mathbb{N},\ 0 \leq \ell < |k|, \tag{2.4}$$

where $u_0, u_1, \ldots, u_{|k|-1} \in \mathbb{K}$ are arbitrary constants.

(c) If $\Omega = \mathbb{Z}$, then the solution u satisfies (2.2) and (2.4) with $u_0, u_1, \ldots, u_{|k|-1} \in \mathbb{K}$ being arbitrary constants.

Proof. The statements (1) and (2) are obvious. Lemma 2.1 implies (3) and (4). We now prove (5). The equation (2.1) can be written as

$$u_{j-k} - z u_j = f_j, \quad j \in \Omega. \tag{2.5}$$

Pick arbitrary $\nu, \ell \in \mathbb{Z}$ with $0 \leq \ell < |k|$ and set $v_{\nu,\ell} = z^\nu u_{\nu k + \ell}$. (2.5) implies

$$v_{\nu-1,\ell} - v_{\nu,\ell} = z^{\nu-1} f_{\nu k + \ell}. \tag{2.6}$$

Summing (2.6) over $\nu = 1, 2, \dots, m$ yields (2.2). If $\Omega = \mathbb{N}$ and $z \neq 0$, then the equation (2.5) for $0 \leq j < k$ is equivalent to (2.3). It is readily verified that if u satisfies the identities in (5a), then it solves (2.1). We have thus shown the statement (5a). Other statements can be proved similarly. □

Let us summarize the consequences of Theorem 2.2 for each class.

Corollary 2.3 (Identity Operator)**.** *For $\Omega = \mathbb{Z}, \mathbb{N}$ and $z \neq 1$, the following identities hold.*

$$\operatorname{Ker}(S_0 - 1) = \mathbb{L}(\Omega, \mathbb{K}), \qquad \operatorname{Ran}(S_0 - 1) = 0,$$
$$\operatorname{Ker}(S_0 - z) = 0, \qquad \operatorname{Ran}(S_0 - z) = \mathbb{L}(\Omega, \mathbb{K}).$$

Corollary 2.4 (Right Shift Operator)**.** *Assume $\Omega = \mathbb{N}$, $k > 0$ and $z \neq 0$. Then the following identities hold.*

$$\operatorname{Ker} S_k = 0, \qquad \operatorname{Ran} S_k = \operatorname{Ran} P_k,$$
$$\operatorname{Ker}(S_k - z) = 0, \qquad \operatorname{Ran}(S_k - z) = \mathbb{L}(\Omega, \mathbb{K}).$$

Corollary 2.5 (Left Shift and Rotation Operator)**.** *Assume either $\Omega = \mathbb{N}$ and $k < 0$ or $\Omega = \mathbb{Z}$ and $k \neq 0$. Then, for any $z \neq 0$, the following identities hold.*

$$\operatorname{Ker} S_k = \operatorname{Ran}(1 - P_{-k}), \quad \operatorname{Ran} S_k = \mathbb{L}(\Omega, \mathbb{K}),$$
$$\operatorname{Ker}(S_k - z) = \left\{ \sum_{j=0}^{|k|-1} u_j \xi_j(z) : u_0, u_1, \dots, u_{|k|-1} \in \mathbb{K} \right\}, \quad \operatorname{Ran}(S_k - z) = \mathbb{L}(\Omega, \mathbb{K}),$$

where $\xi_j(z) \in \mathbb{L}(\Omega, \mathbb{K})$ for $j = 0, 1, \dots, |k| - 1$ is the eigenfunction of S_k associated with eigenvalue z given by

$$(\xi_j(z))_{-mk+\ell} = z^m \delta_{\ell j}, \quad m \in \Omega,\ \ell \in \mathbb{N},\ 0 \leq \ell < |k|.$$

$\{\xi_j(z)\}_j$ is a basis for $\operatorname{Ker}(S_k - z)$.

3. Right Inverse

We now derive a representation of the solutions of (2.1). Suppose $z \neq 0$ and $k \neq 0$. We observe that $S_k - z$ is surjective. It follows that the equation (2.1) is solvable for u. The solution is unique for the right shift operator, but not for the left shift and rotation operator due to $|k|$-dimensional kernel $\operatorname{Ker}(S_k - z)$.

Let us construct a right inverse $R_k(z)$ for $S_k - z$. Suppose $(k, z) \neq (0, 0)$. We define the operator $R_k(z) : \mathbb{L}(\Omega, \mathbb{K}) \longrightarrow \mathbb{L}(\Omega, \mathbb{K})$ by

$$(R_k(z) f)_{mk+\ell} = \frac{1}{z^m} \left([k > 0 \text{ and } \Omega = \mathbb{N}] \frac{f_\ell}{z} + \sum_{j \in \mathbb{Z}} \eta_{m,j} z^{j-1} f_{jk+\ell} \right), \tag{3.1}$$

for $m \in \Omega,\ \ell \in \mathbb{N},\ 0 \leq \ell < |k|$, where the coefficient $\eta_{m,j}$ is defined by

$$\eta_{m,j} = [m < j \leq 0] - [1 \leq j \leq m].$$

For any predicate P, we set $[P] = 1$ if P is true, and $[P] = 0$ otherwise. Direct computation shows the following formulae:

$$\eta_{m-1,j} - \eta_{m,j} = \delta_{jm}, \quad \eta_{m,j+1} - \eta_{m,j} = [m \neq 0](\delta_{jm} - \delta_{j0}). \tag{3.2}$$

Finally, we set $R_k(0)f = S_{-k}f$ for $k \in \mathbb{Z}$, and $R_0(z)f = (1-z)^{-1}f$ for $z \neq 1$. We leave $R_0(z)$ undefined for $z = 1$. The following result can be proved by using the definition of $R_k(z)$ and (3.2).

Theorem 3.1. *Suppose $(k, z) \neq (0, 1)$. Then $R_k(z)$ is a right inverse for $S_k - z$. Namely, we have*

$$(S_k - z)R_k(z)f = f, \quad f \in \mathbb{L}(\Omega, \mathbb{K}).$$

4. Concluding Remarks

We now discuss the commutativity of S_k and the right inverse $R_k(z)$. To this end, let us now briefly show some general properties of a right inverse. Suppose that S is an everywhere defined linear operator on a $\mathbb{K}$-linear space $\mathfrak{X}$. We assume that a right inverse $R : \mathfrak{X} \longrightarrow \mathfrak{X}$ for S exists and that R is everywhere defined. Then we have $SR = 1$. Set $Q = RS$. Then Q is an everywhere defined linear operator on $\mathfrak{X}$. Note that Q is a projection, viz. $Q^2 = Q$. We write $\overline{Q} = 1 - Q$. $\overline{Q}$ is a projection as well. Observe that $SQ = S$ and hence $S\overline{Q} = 0$. It is readily verified that $\operatorname{Ker} S = \operatorname{Ran} \overline{Q}$. We refer to Q as the projection associated with the right inverse R. It is easy to see that S and R commutes if and only if $\operatorname{Ker} S = 0$.

We now return to the case of the shift operator S_k. Set $Q_k(z) = R_k(z)(S_k - z)$. Let u be a solution to (2.1). Then $Q_k(z)u = R_k(z)f$, and hence we can write

$$u = \overline{Q_k(z)}u + R_k(z)f.$$

This means that spectral properties of S_k can be analyzed through the right inverse operator.

In addition, the discussion above on the commutativity indicates that S_k commutes with $R_k(z)$ for the identity and the right shift operator, so $R_k(z)$ is the inverse of $S_k - z$. This is not the case for the left shift and the rotation operator. In the case of the left shift and the rotation, we cannot modify $R_k(z)$ by adding a linear operator $A_k(z)$, in such a way that $R_k(z) + A_k(z)$ is an inverse of $S_k - z$, since $\operatorname{Ker}(S_k - z)$ is non-trivial.

As we have seen above, shift operators are quite simple in definition, but the spectral properties are rather complicated in the infinite dimensional case. It is still an open question to elucidate spectral properties of the shift operators. For example, it is not known whether an invariant subspace decomposition for shift operators is possible or not.

References

[1] Take-Yuki Nagao. Representation of lists by partial functions. *Bulletin of Advanced Institute of Industrial Technology*, 9:155–158, 2015.

（2019.9.20 受稿，2019.10.23 受理）

— Abstract —

A representation of the solution of the inhomogeneous equation is presented, regarding shift operators acting on an infinite dimensional linear space. A unified formulation is proposed to include the identity, right shift, left shift, and rotation operators. An explicit formula is derived for computing a right inverse of the shift operators at all spectral parameters.