A β-core of Incomplete Information Games

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1 Introduction

In the last three decades the economics of information has become one of the major areas of research. These ideas have been applied to the cooperative game theory in recent years such as Yannelis (1991). In this paper we introduce cooperative games derived from a strategic form game with asymmetric information, and prove the existence of its β-core in the TU (Transferable Utility) game.

The related work dates back to the famous economic situation introduced by Scarf (1971). He applied Shapley and Shubik's (1969) approach to n-person normal form games. He derived the NTU (Non Transferable Utility) game from an n-person normal form game, and verified the existence of an α-core, which is considered as a natural extension of core in the strategic game, in the NTU-game. Wilson (1978) added the notion of asymmetric information to this model, and proved the existence of the fine core in asymmetric market game. Yannelis (1991) extended Wilson's model to the strategic form game with continuum states, and proved the non-emptiness of an α-core under uncertainty.

On the contrary, the general existence of β-core, which is also considered as another natural extension of core in the strategic form game, bas been unknown for nearly forty years. Nakayama (1998) and Zhao (1999), however, attempted to show the non-emptiness of β-core in the class of NTU and TU games. They assume respectively that each coalition has a punishment which is called a dominant punishing strategy, and that the characteristic function is strongly separable. They conclude that there exists the β-core, which is identical to an α-core, in the NTU or TU-game to be generated from a normal form game respectively.

In this paper we progress one step ahead of their results. We investigate an incomplete information game instead of an n-person normal form game. In this incomplete information game, each player has different information and a private prior probability. One of the main contributions of our paper is to prove the existence of a β-core in the class of TU-games which are derived from incomplete information games. This result is considered as an asymmetric information version of Nakayama (1998) and Zhao (1999), and a β-core version of Yannelis (1991).

This paper is organized as follows. The next section describes the model. In section 3 we prove our results. Section 4 contains some concluding remarks.

2 Definitions

Let us begin our analysis by describing an incomplete information game to be employed in this paper.

DEFINITION 1: An *incomplete information game* is a system $(N, X_i, u_i, \Omega, \mathcal{F}_i, P)$, where

(1) $N = \{1, \ldots, n\}$ is the set of players,

(2) $(\Omega, \mathcal{F}, P)$ is a probability space,

(3) $\mathcal{F}_i \subset \mathcal{F}$ is a sub σ-algebra,

(4) $X_i = \{x_i \in L^\infty(\Omega, \mathcal{F}, P) \mid x_i : \Omega \to \mathbb{R}$ is $\mathcal{F}_i$-measurable$\}$ is the set of player i's strategies, and

(5) $u_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is player i's $\mathcal{F}_i$-measurable payoff function which is denoted by $u_i(\omega, x)$.

We consider Ω as the set of states. Condition 3 of Definition 1 means that the σ-algebra $\mathcal{F}_i$ is player i's knowledge and the probability measure P is the common prior. As usual, we consider the conditional probability $P(\cdot|\mathcal{F}_i)$ as player i's subjective probability in this model. To simplify notations we define the subjective expected utility function $U_i : \prod_{i=1}^{n} X_i \to \mathbb{R}$, which is described as $U_i(x) = \int_\Omega u_i(\omega, x(\omega))\, dP_i(\omega)$, where $P_i = P(\cdot|\mathcal{F}_i)$. The measurability of condition 4 describes that each player can choose a strategy which he or she knows.

DEFINITION 2 : A TU-game is characterized by a system (N, v), where

(1) $N = \{1, \ldots, n\}$ is the set of players, and

(2) $v : 2^N \setminus \{\emptyset\} \to \mathbb{R}$ is called a characteristic function.

We shall let the subscripts in small letters to denote individual players, and the subscripts in capital letters to denote coalitions. As usual, let $x_S = (x_i \in X_i | i \in S) \in \prod_{i \in s} X_i$ be the strategies of a coalition S and $x_{-S} = (x_i \in X_i | i \in N \setminus S) \in \prod_{i \in N \setminus S} X_i$ be the strategies of the complementary coalition $N \setminus S$. According to Ichiishi (1997), we introduce some behavior rules in this TU-game using an incomplete information game. The characteristic function $v_\beta : 2^N \setminus \{\emptyset\} \to \mathbb{R}$ is called a *β-fashion* in TU-games, if

$$v_\beta(S) = \min_{y_{-S}} \max_{x_S} \sum_{i \in S} U_i(x_s, y_{-s}) = \sum_{i \in S} U_i(x_S^*(\hat{y}_{-s}), \hat{y}_{-s}),$$

where $x_S^*(\hat{y}_{-S})$ is the best response to $\hat{y}_{-S}$. This characteristic function represents an optimistic behavior for coalitions, which players in coalition consider they can alwayes choose the optimail strategy to the opposite strategy for outside coalition.

A TU-game derived from an incomplete information game is called the β-TU game if $v(S) = v_\beta(S)$ for all $S \in 2^N \setminus \{\emptyset\}$.

Definition 3 : The core of the TU-game (N, v) is the set of $u^* \in \mathbb{R}^n$ of a utility vector satisfying

(1) $\sum_{i=1}^{n} u_i^* = v(N)$, and

(2) $\forall S \in 2^N \setminus \{\emptyset\} : \sum_{i \in S} u_i^* \geq v(S)$.

If the characteristic function v is given by v_β, the core of the β-TU game (N, v_β) is called the β-core.

In this paper we prove the non-emptiness of the β-core in the TU-game which is derived from an incomplete information game.

3 Results

Proposition 1: *Let $(N, X_i, u_i, \Omega, \mathcal{F}_i, P)$ be an incomplete information game. Suppose that*

(1) X_i is convex and weakly compact,

(2) for all $\omega \in \Omega$, $u_i(\omega, \cdot)$ is concave on $\mathbb{R}^n$ and u_i is integrably bounded on Ω, and

(3) $u_i(\omega, x_S^*(\hat{y}_{-S})(\omega), \hat{y}_{-S}(\omega)) = \min_{y_{-S}} u_i(\omega, x_S^*(\hat{y}_{-S})(\omega), y_{-S}(\omega))$ for all $\omega \in \Omega$ and $S \in 2^N \setminus \{\emptyset\}$.[(1)]

Then a β-core in the TU-game (N, v_β) is nonempty.

Proof of Proposition 1

The proof is divided into two steps.

Step 1: $U_i : \prod_{i=1}^{n} X_i \to \mathbb{R}$ is a concave and continuous function.

concavity

For all x and y in $\prod_{i=1}^{n} X_i$ and for all $t \in (0, 1)$,

(1) This condition is considered as an incomplete information version of Zhao's (1999b) strong separability. It signifies that the outsider's action which best punishes a coalition S at a state ω is also the action which best punishes each member of the coalition at a state ω. Nakayama (1998) also proposed a related condition, which was called a dominant punishing strategy, for an existence of a β-core in NTU-games. It means that each coalition has a punishment strategy to its opposite coalition. It is known that this condition is a little stronger than the previous one.

$$U_i(tx+(1-t)y)=\int_\Omega u_i(\omega, tx(\omega)+(1-t)y(\omega))\,dP_i(\omega)$$

$$\geq \int_\Omega tu_i(\omega, x(\omega))+(1-t)u_i(\omega, y(\omega))\,dP_i(\omega)$$

$$(u_i \textit{ is concave})$$

$$= tU_i(x)+(1-t)U_i(y).$$

continuity

Noting that u_i is continuous on the interior of its domain because of the concavity of u_i on $\mathbb{R}^n$. Suppose that $x_n \to x$ in $\prod_{i=1}^n X_i$. Since $\|x_n - x\|_\infty = \text{ess sup}_\omega |x_n(\omega) - x(\omega)|$, it follows that $x_n(\omega) \to x(\omega)$ almost all ω. This implies that $u_i(\omega, x_n(\omega)) \to u_i(\omega, x(\omega))$ almost all ω.

$$\lim_{n\to\infty} U_i(x_n)=\lim_{n\to\infty}\int_\Omega u_i(\omega, x_n(\omega))dP_i(\omega)$$

$$=\int_\Omega \lim_{n\to\infty} u_i(\omega, x_n(\omega))\,dP_i(\omega) \qquad \text{(DCT)}$$

$$=\int_\Omega u_i(\omega, x(\omega))\,dP_i(\omega) \qquad (u_i \text{ is continuous})$$

$$= U_i(x)$$

Step 2: $\forall S \in 2^N \setminus \{\emptyset\}: \sum_{S\in\mathcal{B}} w_S v_\beta(S) \leq v_\beta(N)$ for all balanced collection $\mathcal{B}$

Before proving, it is convenient to introduce the following balanced collection. Let $\mathcal{B} = \{T_1, ..., T_k\}$ be a collection of coalitions. For each $i \in T$, $\mathcal{B}(\mathrm{i}) = \{T \in \mathcal{B} \mid i \in T\}$ denotes the set of these coalitions of which i is a member. If there is nonnegative number w_T for all $T \in \mathcal{B}$ such that $\sum_{T\in\mathcal{B}(i)} w_T = 1$ for all $i \in N$, $\mathcal{B}$ is called a balanced collection. It has been known by Bondareva-Shapley that the non-emptiness of the core in TU-games is equivalent to the balanced condition, that is $\forall S \in 2^N \setminus \{\emptyset\}: \sum_{S\in\mathcal{B}} w_S v_\beta(S) \leq v_\beta(N)$ for all balanced collection $\mathcal{B}$.

It follows that $\min_{y_{-s}} \sum_{i\in S} U_i(x_S^*(y_{-S}), y_{-S})$ is lower semi-continuous, since X_i is weakly compact and U_i is a continuous function. For this reason, v_β is well defined.

Noting that $v_\beta(S)$ is given by $\sum_{i\in S} U_i(x_S^*(\hat{y}_{-S}), \hat{y}_{-S})$, where $x_S^*(\hat{y}_{-S})$ is the best response to $\hat{y}_{-S}$, we check the balanced condition of Bondareva-Shapley.

$$\sum_{S\in\mathcal{B}} w_S v_\beta(S) = \sum_{S\in\mathcal{B}} w_S \sum_{i\in S} U_i(x_S^*(\hat{y}_{-S}), \hat{y}_{-S})$$

$$=\sum_{i=1}^n \sum_{S\in\mathcal{B}(i)} w_S U_i(x_S^*(\hat{y}_{-S}), \hat{y}_{-S})$$

$$=\sum_{i=1}^n \sum_{S\in\mathcal{B}(i)} w_S \int_\Omega u_i(\omega, x_S^*(\hat{y}_{-S})(\omega), \hat{y}_{-S}(\omega)\,dP_i(\omega)$$

$$\leq \sum_{i=1}^{n} \sum_{S \in \mathcal{B}(i)} w_S \int_{\Omega} u_i(\omega, x_S^*(\hat{y}_{-S})(\omega), z_{-S}(\omega))\, \mathrm{dPi}(\omega)$$

$$= \sum_{i=1}^{n} \sum_{S \in \mathcal{B}(i)} w_S U_i(x_S^*(\hat{y}_{-S}), z_{-S})$$

$$\leq \sum_{i=1}^{n} U_i(\sum_{S \in \mathcal{B}(i)} w_S (x_S^*(\hat{y}_{-S}), z_{-S})) \qquad \text{(the concavity of } U_i\text{)}$$

$$\leq \max_{x \in \prod_{i=1}^{n} X_i} \sum_{i=1}^{n} U_i(x) = v_\beta(N).$$

Then, we can conclude the nonemptiness of β-core from Bondareva-Shapley's theorem.

4 Concluding Remarks

We have established the existence of a β-core of an incomplete information game. The non-emptiness of an β-core requires (1) a payoff function is concave, and (2) player i's strategy is $\mathcal{F}_i$-measurable, and its set is convex and weakly compact (3) vb is the outside minimum strategy condition. This result may be considered as an asymmetric information version of Scarf (1971) and Zhao (1999a) . Our future topic is to introduce a learning model. To do so, we may analyze a selection of core allocations and a path of a core allocation.

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―Abstract―

We prove an existence of an α-core and a β-core derived from an incomplete information game in which each player has different information. These results can be considered as an asymmetric information version of Zhao (1999).