

An α -core of an Incomplete Information Game

UTSUMI, Yukihiisa

1. Introduction

In the last two decades the economics of information has become one of the major areas of research. These ideas have been applied to the cooperative game theory in recent years. In this paper we introduce asymmetric information and prove the existence of an α -core in the TU (Transferable Utility) game derived from an incomplete information game.

The related work dates back to the article by Scarf (1971). He applied Shapley and Shubik's (1969) approach to n-person normal form games. He derived the NTU (Non Transferable Utility) game from an n-person normal form game, and verified the existence of an α -core in the NTU-game. Wilson (1978) added the notion of different information or uncertainty to this model and showed the same result. Yannelis (1991) extended Wilson's model to the infinitely many (continuum of) states, and proved the non-emptiness of an α -core under uncertainty.

In this paper we progress one step ahead of their results. We investigate an incomplete information game instead of an n-person normal form game. In this incomplete information game, each player has different information and a private prior probability. One of the main contributions of our paper is to prove the existence of an α -core in the class of TU-games which are derived from incomplete information games. This result is considered as an asymmetric information version of Zhao (1999a).

This paper is organized as follows. The next section describes the model. In section 3 we prove our main results. Section 4 contains some concluding remarks.

2. Definitions

Let us begin our analysis by describing an incomplete information game to be employed in this paper.

DEFINITION 1: An *incomplete information game* is a system $(N, X_i, u_i, \Omega, \mathcal{F}_i, P)$, where

- (1) $N = \{1, \dots, n\}$ is the set of players,
- (2) (Ω, \mathcal{F}, P) is a probability space,

- (3) $\mathcal{F}_i \subset \mathcal{F}$ is a sub σ -algebra,
- (4) $X_i = \{x_i \in L^\infty(\Omega, \mathcal{F}, P) \mid x_i : \Omega \rightarrow \mathbb{R} \text{ is } \mathcal{F}_i\text{-measurable}\}$ is the set of player i 's strategies, and
- (5) $u_i : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ is player i 's \mathcal{F}_i -measurable payoff function which is denoted by $u_i(w, x)$

We consider Ω as the set of states. Condition 3 of Definition 1 means that the σ -algebra \mathcal{F}_i is player i 's knowledge and the probability measure P is the common prior. As usual, we consider the conditional probability $P(\cdot \mid \mathcal{F}_i)$ as player i 's subjective probability in this model. To simplify notations we define the subjective expected utility function $U_i : \prod_{i=1}^n X_i \rightarrow \mathbb{R}$, which is described as $U_i(x) = \int_{\Omega} u_i(w, x(w)) dP_i(w)$, where $P_i = P(\cdot \mid \mathcal{F}_i)$. The measurability of condition 4 describes that each player can choose a strategy which he or she knows.

DEFINITION 2: A TU-game is characterized by a system (N, v) , where

- (1) $N = \{1, \dots, n\}$ is the set of players, and
- (2) $v : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}$ is called a characteristic function.

We shall let the subscripts in small letters to denote individual players, and the subscripts in capital letters to denote coalitions. As usual, let $x_S = (x_i \in X_i \mid i \in S) \in \prod_{i \in S} X_i$ be the strategies of a coalition S and $x_{-S} = (x_i \in X_i \mid i \in N \setminus S) \in \prod_{i \in N \setminus S} X_i$ be the strategies of the complementary coalition $N \setminus S$. According to Ichiishi (1997), we introduce some behavior rules in this TU-game using an incomplete information game. The characteristic function $v_\alpha : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}$ is called an α -fashion in TU-games, if

$$v_\alpha(S) = \max_{x_S} \min_{y_{-S}} \sum_{i \in S} U_i(x_S, y_{-S}) = \sum_{i \in S} U_i(\hat{x}_S, y_{-S}^*(\hat{x}_S)),$$

where $y_{-S}^*(\hat{x}_S)$ is the best punishment to \hat{x}_S .

A TU-game derived from an incomplete information game is defined as $v(S) = v_\alpha(S)$ for all $S \in 2^N \setminus \{\emptyset\}$.

DEFINITION 3: The α -core of the TU-game (N, v_α) is the set of utility vector $u^* \in \mathbb{R}^n$ satisfying

- (1) $\sum_{i=1}^n u_i^* = v_\alpha(N)$, and
- (2) $\sum_{i \in S} u_i^* \geq v_\alpha(S)$ for all $S \in 2^N \setminus \{\emptyset\}$

In this paper we prove the non-emptiness of the α -core in the TU-game which is derived from an incomplete information game.

3. Results

PROPOSITION 1: *Let $(N, X_i, u_i, \Omega, \mathcal{F}_i, P)$ be an incomplete information game. If $X_i \subset L^\infty(\Omega, \mathcal{F}, P)$ is convex and weakly compact, and u_i is concave on \mathbb{R}^n and integrably bounded function, then an α -core in the TU-game (N, v_α) is nonempty.*

Proof of Proposition 1

The proof is divided into two steps.

Step 1: $U_i : \prod_{i=1}^n X_i \rightarrow \mathbb{R}$ is a concave and continuous function.

concavity

For all x and y in $\prod_{i=1}^n X_i$ and for all $t \in (0, 1)$,

$$\begin{aligned} U_i(tx + (1-t)y) &= \int_{\Omega} u_i(w, tx(w) + (1-t)y(w)) dP_i(w) \\ &\geq \int_{\Omega} tu_i(w, tx(w)) + (1-t)u_i(w, y(w)) dP_i(w) \\ &\qquad\qquad\qquad (u_i \text{ is concave}) \\ &= tU_i(x) + (1-t)U_i(y). \end{aligned}$$

continuity

Noting that u_i is continuous on the interior of its domain because of the concavity of u_i on \mathbb{R}^n . Suppose that $x_n \rightarrow x$ in $\prod_{i=1}^n X_i$. Since $\|x_n - x\|_\infty = \text{ess sup}_w |x_n(w) - x(w)|$, it follows that $x_n(w) \rightarrow x(w)$ almost all w . This implies that $u_i(w, x_n(w)) \rightarrow u_i(w, x(w))$ almost all w . Using the dominated convergence theorem, we obtain the continuity of U_i . In fact,

$$\begin{aligned} \lim_{n \rightarrow \infty} U_i(x_n) &= \lim_{n \rightarrow \infty} \int_{\Omega} u_i(w, x_n(w)) dP_i(w) \\ &= \int_{\Omega} \lim_{n \rightarrow \infty} u_i(w, x_n(w)) dP_i(w) && \text{(DCT)} \\ &= \int_{\Omega} u_i(w, x(w)) dP_i(w) && (U_i \text{ is continuous}) \\ &= U_i(x). \end{aligned}$$

Step 2: $\sum_{S \in B} w_s v_\alpha(S) \leq v_\alpha(N)$ for all $S \in 2^N \setminus \{\emptyset\}$ and all balanced collection B

Before proving, it is convenient to introduce the following balanced collection. Let $B = \{T_1, \dots, T_k\}$ be a collection of coalitions. For each $i \in T$, $B(i) = \{T \in B \mid i \in T\}$

denotes the set of these coalitions of which i is a member. If there is nonnegative number w_T for all $T \in B$ such that $\sum_{T \in B(i)} w_T = 1$ for all $i \in N$, B is called a balanced collection. It has been known by Bondareva-Shapley that the non-emptiness of the core in TU-games is equivalent to the balanced condition, that is $\sum_{S \in B} w_S v_\beta(S) \leq v_\beta(N)$ for all $S \in 2^N \setminus \{\emptyset\}$ and all balanced collection B .

It follows that $\max_{x_s} \sum_{i \in S} U_i(x_s, y_{-s}^*(x_s))$ is lower semi-continuous, since X_i is weakly compact and U_i is a continuous function. For this reason, v_β is well defined.

Noting that $v_\alpha(S)$ is given by $\sum_{i \in S} U_i(\hat{x}_s, y_{-s}^*(\hat{x}_s))$, where $y_{-s}^*(\hat{x}_s)$ is the best punishment to \hat{x}_s , we check the balanced condition of Bondareva-Shapley.

Since \hat{y}_{-s} is the best punishment to \hat{x}_s , we obtain for all $z_{-s} \in \prod_{i \in N \setminus S} X_i$

$$\begin{aligned} \sum_{S \in B} w_S v_\alpha(S) &= \sum_{S \in B} w_S \sum_{i \in S} U_i(\hat{x}_s, y_{-s}^*(\hat{x}_s)) \\ &\leq \sum_{S \in B} w_S \sum_{i \in S} U_i(\hat{x}_s, z_{-s}) \end{aligned}$$

Using the concavity of U_i , we obtain that

$$\begin{aligned} \sum_{S \in B} w_S \sum_{i \in S} U_i(\hat{x}_s, z_{-s}) &= \sum_{i=1}^n \sum_{S \in B(i)} w_S U_i(\hat{x}_s, z_{-s}) \\ &\leq \sum_{i=1}^n U_i\left(\sum_{S \in B(i)} w_S \hat{x}_s, z_{-s}\right) \\ &\leq \max_{x \in \prod_{i=1}^n X_i} \sum_{i=1}^n U_i(x) = v_\alpha(N). \end{aligned}$$

Then we can conclude from Bondareva-Shapley's theorem that the α -core is nonempty.

4. Concluding Remarks

We have established the existence of an α -core of an incomplete information game. The non-emptiness of an α -core requires (1) a payoff function is concave, and (2) player i 's strategy is \mathcal{F}_i -measurable, and its set is convex and weakly compact. This result may be considered as an asymmetric information version of Scarf (1971) and Zhao (1999a). Our future topic is to prove the existence of β -core of an incomplete information game.

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— Abstract —

We prove an existence of an α -core derived from an incomplete information game in which each player has different information. These results can be considered as an asymmetric information version of Zhao (1999a).