Solution of Gyldén-Lindstedt Equation

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Abstract

It often occurs in the celestial mechanics a linear differential equation with a periodic coefficient

$$\frac{d^2x}{dt^2} + m^2x = 2\varepsilon\, x \cos w', \quad w' = \lambda\, t + \beta,$$

where m, λ, β are given constants and ε is a small given constant.

In the present paper I give a solution and its mean motion of the above equation up to the 6th order of ε by making use of Lindtstedt's method which does not give rise to mixed secular terms.

1 Gyldén-Lindstedt Equation

In the present paper we consider a linear differential equation

$$\frac{d^2x}{dt^2} + m^2x = 2\varepsilon\, x \cos w', \quad w' = \lambda\, t + \beta, \quad (\,\varepsilon : \text{ small parameter}\,) \tag{1}$$

It is a particular case of a linear differential equation with periodic coefficients

$$\frac{d^2x}{dt^2} + x(q^2 + 2q_1 \cos 2t + 2q_2 \cos 4t + \cdots) = 0$$

The particular equation (1) and the general equation were studied by Gyldén ([1], [2]), Lindstedt ([3]~[[6]), Bruns ([7]), Callandreau ([8]) and others ([9], [10], [11]). As the equation (1) was first discussed by Gyldén and Lindstedt, Tisserand named it Gyldén-Lindstedt Equation.

2 Solution having mixed secular terms

In order to solve Gyldén-Lindstedt equation we put

$$x = x_0 + \varepsilon\, x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots \tag{2}$$

and substitute it in equation (1). By equating terms of the same order of ε, we get

$$\begin{aligned} \ddot{x}_0 + m^2 x_0 &= 0, && (3)\\ \ddot{x}_1 + m^2 x_1 &= 2\, x_0 \cos w', && (4)\\ \ddot{x}_2 + m^2 x_2 &= 2\, x_1 \cos w', && (5)\\ \ddot{x}_3 + m^2 x_3 &= 2\, x_2 \cos w', && (6)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

2.1 0th order solution

From (3) we obtain

$$x_0 = \eta_0 \cos w, \quad w = m\, t + \alpha\,, \tag{7}$$

where η_0 and α are integral constants.

2.2 1st order solution

By substitution of (7) in the right hand side of (4), we get

$$\text{r.h.s.} = 2\, x_0 \cos w' = \eta_0 \cos(w + w') + \eta_0 \cos(w - w')\,;$$

by putting $x_1 = \eta_0\big\{B_1 \cos(w + w') + C_1 \cos(w - w')\big\}$, l.h.s. of (4) gives

$$\begin{aligned} \ddot{x}_1 =& \eta_0\big\{-(m+\lambda)^2 B_1 \cos(w + w') - (m-\lambda)^2 C_1 \cos(w - w')\big\},\\ m^2 x_1 =& \eta_0\big\{m^2 B_1 \cos(w + w') + m^2 C_1 \cos(w - w')\big\} \end{aligned}$$

Equating coefficients of $\cos(w + w')$ and $\cos(w - w')$ of both sides respectively, we get following equations from which we can determine A_1 and B_1 as functions of m and λ :

$$\big\{m^2 - (m+\lambda)^2\big\} B_1 = 1, \quad \therefore B_1 = \frac{1}{m^2 - (m+\lambda)^2}\,, \tag{8}$$

$$\big\{m^2 - (m-\lambda)^2\big\} C_1 = 1, \quad \therefore C_1 = \frac{1}{m^2 - (m-\lambda)^2}\,. \tag{9}$$

Therefore we obtain as a first order solution

$$x_1 = \eta_0 \left\{ \frac{1}{m^2 - (m+\lambda)^2} \cos(w + w') + \frac{1}{m^2 - (m-\lambda)^2} \cos(w - w') \right\} \tag{10}$$

2.3 2nd order solution

By substitution of (10) in the right hand side of (5), we get

$$\text{r.h.s.} = 2\, x_1 \cos w' = \eta_0\big\{B_1 \cos(w + 2w') + B_1 \cos w + C_1 \cos w + C_1 \cos(w - 2w')\big\}\,;$$

by putting $x_2 = \eta_0\{A_2\, t \sin w + B_2 \cos(w+2w') + C_2 \cos(w-2w')\}$, l.h.s. of (5) gives

$$\begin{aligned}
\ddot{x}_2 =& \eta_0\big\{2\,mA_2 \cos w - m^2 A_2\, t \sin w \\
&\quad - (m+2\lambda)^2 B_2 \cos(w+2w') - (m-2\lambda)^2 C_2 \cos(w-2w')\big\}, \\
m^2 x_2 =& \eta_0\big\{m^2 A_2\, t \sin w + m^2 B_2 \cos(w+2w') + m^2 C_2 \cos(w-2w')\big\}
\end{aligned}$$

Adding the above two equations ($= \ddot{x}_2 + m^2 x_2$), we see that a mixed secular term $t \sin w$ vanishes. Equating coefficients of $\cos w, \cos(w+2w')$ and $\cos(w-2w')$ of both sides respectively, we get following equations

$$\begin{aligned}
2\,mA_2 = B_1 + C_1 &= \frac{1}{m^2-(m+\lambda)^2} + \frac{1}{m^2-(m-\lambda)^2}, \\
\big\{m^2-(m+2\lambda)^2\big\}\, B_2 = B_1 &= \frac{1}{m^2-(m+\lambda)^2}, \\
\big\{m^2-(m-2\lambda)^2\big\}\, C_2 = C_1 &= \frac{1}{m^2-(m-\lambda)^2},
\end{aligned}$$

From these equations A_2, B_2, C_2 can be detemined as functions of m and λ :

$$A_2 = \frac{1}{2m}(B_1+C_1) = \frac{1}{2m}\left\{\frac{1}{m^2-(m+\lambda)^2} + \frac{1}{m^2-(m-\lambda)^2}\right\} \tag{11}$$

$$B_2 = \frac{1}{m^2-(m+2\lambda)^2}\, B_1 = \frac{1}{\{m^2-(m+\lambda)^2\}\{m^2-(m+2\lambda)^2\}} \tag{12}$$

$$C_2 = \frac{1}{m^2-(m-2\lambda)^2}\, C_1 = \frac{1}{\{m^2-(m-\lambda)^2\}\{m^2-(m-2\lambda)^2\}} \tag{13}$$

Finally we obtain as a second order solution

$$\begin{aligned}
x_2 =& \eta_0\left\{-\frac{1}{m(\lambda^2-4m^2)}\, t \sin w \right.\\
&+ \frac{1}{\{m^2-(m+\lambda)^2\}\{m^2-(m+2\lambda)^2\}} \cos(w+2w') \\
&\left.+ \frac{1}{\{m^2-(m-\lambda)^2\}\{m^2-(m-2\lambda)^2\}} \cos(w-2w')\right\}
\end{aligned} \tag{14}$$

2.4 3rd order solution

Substitution of (14) in the right hand side of (6) gives

$$\begin{aligned}
\text{r.h.s.} =& \eta_0\Big[A_2\, t\,\big\{\sin(w+w') + \sin(w-w')\big\} \\
&+ B_2\big\{\cos(w+3w') + \cos(w+w')\big\} + C_2\big\{\cos(w-w') + \cos(w-3w')\big\}\Big]
\end{aligned}$$

By putting

$$x_3 = \eta_0\big[A_3'\, t \sin(w+w') + A_3''\, t \sin(w-w')$$

$$+ B_3' \cos(w + w') + B_3'' \cos(w + 3w') + C_3' \cos(w - w') + C_3'' \cos(w - 3w')\big]$$

the left hand side of (6) gives

$$\begin{aligned}
\ddot{x}_3 =& \eta_0 \big\{ -(m+\lambda)^2 A_3' \, t \sin(w+w') - (m-\lambda)^2 A_3'' \, t \sin(w-w') \\
& + 2(m+\lambda) A_3' \cos(w+w') + 2(m-\lambda) A_3'' \cos(w-w') \\
& - (m+\lambda)^2 B_3' \cos(w+w') - (m+3\lambda)^2 B_3'' \cos(w+3w') \\
& - (m-\lambda)^2 C_3' \cos(w-w') - (m-3\lambda)^2 C_3'' \cos(w-3w') \big\} \\
m^2 x_3 =& \eta_0 \big\{ m^2 A_3' \, t \sin(w+w') + m^2 A_3'' \, t \sin(w-w') \\
& + m^2 B_3' \cos(w+w') + m^2 B_3'' \cos(w+3w') \\
& + m^2 C_3' \cos(w-w') + m^2 C_3'' \cos(w-3w') \big\}
\end{aligned}$$

Comparing coefficients of both sides, we get following equtions from which we can determine $A_3', A_3'', B_3', B_3'', C_3', C_3''$ as functions of m and λ :

$$\{m^2 - (m+\lambda)^2\} A_3' = A_2, \quad \therefore A_3' = \frac{A_2}{m^2 - (m+\lambda)^2} \tag{15}$$

$$\{m^2 - (m-\lambda)^2\} A_3'' = A_2, \quad \therefore A_3'' = \frac{A_2}{m^2 - (m-\lambda)^2} \tag{16}$$

$$2(m+\lambda) A_3' + \{m^2 - (m+\lambda)^2\} B_3' = B_2, \quad \therefore B_3' = \frac{B_2 - 2(m+\lambda) A_3'}{m^2 - (m+\lambda)^2} \tag{17}$$

$$2(m-\lambda) A_3'' + \{m^2 - (m-\lambda)^2\} C_3' = C_2, \quad \therefore C_3' = \frac{C_2 - 2(m+\lambda) A_3''}{m^2 - (m-\lambda)^2} \tag{18}$$

$$\{m^2 - (m+3\lambda)^2\} B_3'' = B_2, \quad \therefore B_3'' = \frac{B_2}{m^2 - (m+3\lambda)^2} \tag{19}$$

$$\{m^2 - (m-3\lambda)^2\} C_3'' = C_2, \quad \therefore C_3'' = \frac{C_2}{m^2 - (m-3\lambda)^2} \tag{20}$$

Finally we get a third order solution which is not explicitly written here.

3 Lindstedt's Method of Solution — Solution not having mixed secular terms

The solution of Gyldén-Lindstet equation (1) obtained in the previous section had mixed secular terms such as $t \sin w$ in x_2 and $t \sin(w+w'), t \sin(w-w')$ in x_3.

The reason $t \sin w$ appears in x_2 is that a term $\cos w$ occurs in the right hand side of (5). So in order that $\cos w$ does not occur we slightly change (1) into

$$\ddot{x} + m^2(1-\nu) = -m^2 \nu x + 2\varepsilon x \cos w' \tag{21}$$

If we assume here

$$n^2 = m^2(1-\nu) \tag{22}$$

(21) becomes as follows :

$$\ddot{x} + n^2 x = -m^2 \nu x + 2\varepsilon x \cos w' \tag{23}$$

As secular terms appear in higher order terms than the first order of ε, ν of the right hand side of (23) stars with the second order of ε :

$$\nu = \varepsilon^2 \nu_2 + \varepsilon^3 \nu_3 + \cdots\cdots \tag{24}$$

If σ is defined as follows

$$n \equiv m(1-\sigma) = m(1-\nu)^{\frac{1}{2}} \tag{25}$$

σ is expressed as a series of ε

$$\sigma = \frac{1}{2}\varepsilon^2 \nu_2 + \frac{1}{2}\varepsilon^3 \nu_3 + \frac{1}{8}\varepsilon^4(\nu_2^2 + 4\nu_4) + \cdots \tag{26}$$

In order to obtain expressions corresponding to (3) $\sim$ (6), we substitute (2) into (23) (at this stage we do not develop n of l.h.s. with respect to ε)

$$\ddot{x}_0 + n^2 x_0 = 0, \tag{27}$$
$$\ddot{x}_1 + n^2 x_1 = 2x_0 \cos w', \tag{28}$$
$$\ddot{x}_2 + n^2 x_2 = -m^2 \nu_2 x_0 + 2x_1 \cos w', \tag{29}$$
$$\ddot{x}_3 + n^2 x_3 = -m^2(\nu_3 x_0 + \nu_2 x_1) + 2x_2 \cos w', \tag{30}$$
$$\ddot{x}_4 + n^2 x_4 = -m^2(\nu_4 x_0 + \nu_3 x_1 + \nu_2 x_2) + 2x_3 \cos w', \tag{31}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

3.1 0th and 1st order solution

As (27) and (28) are of the same form with (3) and (4), we get solutions of x_0 and x_1 by substituting m for n in (7) and (10) (we remark that $n = m$ in the case of solutions of 0th and 1st order)

$$x_0 = \eta_0 \cos w, \quad w = nt + \alpha \tag{32}$$
$$x_1 = \eta_0\{B_1^{(1)} \cos(w + w') + C_1^{(1)} \cos(w - w')\} \tag{33}$$

where

$$B_1^{(1)} = \frac{1}{n^2 - (n+\lambda)^2} = -\frac{1}{\lambda(\lambda + 2n)}, \quad C_1^{(1)} = \frac{1}{n^2 - (n-\lambda)^2} = -\frac{1}{\lambda(\lambda - 2n)} \tag{34}$$

3.2 2nd order solution

Substitution of (32) and (33) in the right hand side of (29) gives

$$\text{r.h.s.} = \eta_0\{-m^2 \nu_2 \cos w + B_1^{(1)}\{\cos(w + 2w') + \cos w\} + C_1^{(1)}\{\cos w + \cos(w - 2w')\}\}$$

$$= \eta_0\{(-m^2\nu_2 + B_1^{(1)} + C_1^{(1)})\cos w + B_1^{(1)}\cos(w+2w') + C_1^{(1)}\cos(w-2w')\} \tag{35}$$

If we put

$$m^2\,\nu_2 = B_1^{(1)} + C_1^{(1)} = -\frac{2}{\lambda^2 - 4n^2} \tag{36}$$

a term of $\cos w$ vanishes in the right hand side of (29), which becomes

$$\text{r.h.s. of (29)} = \eta_0\big\{B_1^{(1)}\cos(w+2w') + C_1^{(1)}\cos(w-2w')\big\} \tag{37}$$

Now we put

$$x_2 = \eta_0\big\{B_2^{(2)}\cos(w+2w') + C_2^{(2)}\cos(w-2w')\big\} \tag{38}$$

by which the left hand side of (29) gives

$$\ddot{x}_2 = \eta_0\big\{-(n+2\lambda)^2 B_2^{(2)}\cos(w+2w') - (n-2\lambda)^2 C_2^{(2)}\cos(w-2w')\big\}$$
$$n^2 x_2 = \eta_0\big\{n^2 B_2^{(2)}\cos(w+2w') + n^2 C_2^{(2)}\cos(w-2w')\big\}$$

Comparson of coefficients of the both sides gives

$$\{n^2 - (n+2\lambda)^2\}B_2^{(2)} = B_1^{(1)}$$
$$\{n^2 - (n-2\lambda)^2\}C_2^{(2)} = C_1^{(1)}$$

from which $B_2^{(2)}, C_2^{(2)}$ are determined as follows :

$$B_2^{(2)} = \frac{B_1^{(1)}}{n^2 - (n+2\lambda)^2} = \frac{1}{4\lambda^2(\lambda+n)(\lambda+2n)} \tag{39}$$

$$C_2^{(2)} = \frac{C_1^{(1)}}{n^2 - (n-2\lambda)^2} = \frac{1}{4\lambda^2(\lambda-n)(\lambda-2n)} \tag{40}$$

3.3 3rd order solution

Next we put (32), (33) and (38) in the right hand side of (30), then it becomes

$$\text{r.h.s. of (30)} = \eta_0\Big[-m^2\big[\nu_3\cos w + \nu_2\{B_1^{(1)}\cos(w+w') + C_1^{(1)}\cos(w-w')\}\big]$$
$$+ B_2^{(2)}\{\cos(w+3w') + \cos(w+w')\} + C_2^{(2)}\{\cos(w-w') + \cos(w-3w')\}\Big]$$

In order not to occur a term of $\cos w$ in the above equation, we take

$$\nu_3 = 0\,; \tag{41}$$

by which the right hand side of (30) becomes

$$\text{r.h.s. of (30)} = \eta_0\Big[-m^2\nu_2\big\{B_1^{(1)}\cos(w+w') + C_1^{(1)}\cos(w-w')\big\}$$
$$+ B_2^{(2)}\big\{\cos(w+3w') + \cos(w+w')\big\} + C_2^{(2)}\big\{\cos(w-w') + \cos(w-3w')\big\}\Big]$$

Now we put

$$x_3 = \eta_0 \Big[B_3^{(1)} \cos(w+w') + B_3^{(3)} \cos(w+3w') + C_3^{(1)} \cos(w-w') + C_3^{(3)} \cos(w-3w') \Big] \quad (42)$$

then the left hand side of (30) gives

$$\begin{aligned} \ddot{x}_3 =& \eta_0 \{ -(n+\lambda)^2 B_3^{(1)} \cos(w+w') - (n+3\lambda)^2 B_3^{(3)} \cos(w+3w') \\ & - (n-\lambda)^2 C_3^{(1)} \cos(w-w') - (n-3\lambda)^2 C_3^{(3)} \cos(w-3w') \} \\ n^2 x_3 =& \eta_0 \{ n^2 B_3^{(1)} \cos(w+w') + n^2 B_3^{(3)} \cos(w+3w') \\ & + n^2 C_3^{(1)} \cos(w-w') + n^2 C_3^{(3)} \cos(w-3w') \} \end{aligned}$$

Comparison of coefficients of the both sides gives the following equations

$$\{n^2 - (n+\lambda)^2\} B_3^{(1)} = -m^2 \nu_2 B_1^{(1)} + B_2^{(2)}, \quad \{n^2 - (n+3\lambda)^2\} B_3^{(3)} = B_2^{(2)}$$

$$\{n^2 - (n-\lambda)^2\} C_3^{(1)} = -m^2 \nu_2 C_1^{(1)} + C_2^{(2)}, \quad \{n^2 - (n-3\lambda)^2\} C_3^{(3)} = C_2^{(2)}$$

from these equations

$$B_3^{(1)} = \frac{-m^2 \nu_2 B_1^{(1)} + B_2^{(2)}}{n^2 - (n+\lambda)^2}, \qquad B_3^{(3)} = \frac{B_2^{(2)}}{n^2 - (n+3\lambda)^2}$$

$$C_3^{(1)} = \frac{-m^2 \nu_2 C_1^{(1)} + C_2^{(2)}}{n^2 - (n-\lambda)^2}, \qquad C_3^{(3)} = \frac{C_2^{(2)}}{n^2 - (n-3\lambda)^2}$$

Finally we obtain

$$B_3^{(1)} = \frac{7\lambda^2 + 8n\lambda + 4n^2}{4\lambda^3 (\lambda+n)(\lambda+2n)^3 (\lambda-2n)} \quad (43)$$

$$C_3^{(1)} = \frac{7\lambda^2 - 8n\lambda + 4n^2}{4\lambda^3 (\lambda-n)(\lambda-2n)^3 (\lambda+2n)} \quad (44)$$

$$B_3^{(3)} = -\frac{1}{12\lambda^3 (\lambda+n)(\lambda+2n)(3\lambda+2n)} \quad (45)$$

$$C_3^{(3)} = -\frac{1}{12\lambda^3 (\lambda-n)(\lambda-2n)(3\lambda-2n)} \quad (46)$$

3.4 4th order solution

In order to obtain x_4, we take in the right hand side of (31)

$$m^2 \nu_4 = B_3^{(1)} + C_3^{(1)} = \frac{7\lambda^2 + 20n^2}{2(\lambda^2 - n^2)(\lambda^2 - 4n^2)^3}; \quad (47)$$

then a term of $\cos w$ does not occur in it. Now we put

$$x_4 = \eta_0 \{ B_4^{(2)} \cos(w+2w') + B_4^{(4)} \cos(w+4w') + C_4^{(2)} \cos(w-2w') + C_4^{(4)} \cos(w-4w') \} \quad (48)$$

then we get

$$B_4^{(2)} = \frac{-m^2\nu_2 B_2^{(2)} + B_3^{(1)} + B_3^{(3)}}{n^2-(n+2\lambda)^2}, \quad B_4^{(4)} = \frac{B_3^{(3)}}{n^2-(n+4\lambda)^2}$$
$$C_4^{(2)} = \frac{-m^2\nu_2 C_2^{(2)} + C_3^{(1)} + C_3^{(3)}}{n^2-(n-2\lambda)^2}, \quad C_4^{(4)} = \frac{C_3^{(3)}}{n^2-(n-4\lambda)^2}$$

from which we obtain coefficients $B_4^{(2)}, B_4^{(4)}, C_4^{(2)}$ and $C_4^{(4)}$ as functions of n and λ :

$$B_4^{(2)} = -\frac{5\lambda^2+5n\lambda+2n^2}{3\lambda^4(\lambda+n)(\lambda+2n)^3(\lambda-2n)(3\lambda+2n)} \tag{49}$$
$$C_4^{(2)} = -\frac{5\lambda^2-5n\lambda+2n^2}{3\lambda^4(\lambda-n)(\lambda-2n)^3(\lambda+2n)(3\lambda-2n)} \tag{50}$$
$$B_4^{(4)} = \frac{1}{96\lambda^4(\lambda+n)(\lambda+2n)(2\lambda+n)(3\lambda+2n)} \tag{51}$$
$$C_4^{(4)} = \frac{1}{96\lambda^4(\lambda-n)(\lambda-2n)(2\lambda-n)(3\lambda-2n)} \tag{52}$$

3.5 5th order solution

An equation determing x_5 is

$$\ddot{x}_5 + n^2 x_5 = -m^2(\nu_5 x_0 + \nu_4 x_1 + \nu_3 x_2 + \nu_2 x_3) + 2\,x_4 \cos w'\,; \tag{53}$$

in the right hand side of this equation, terms of ν_5 and x_0 generate a term of $\cos w$. In order not to do so, we take

$$\nu_5 = 0 \tag{54}$$

Now we put

$$x_5 = \eta_0 \{ B_5^{(1)} \cos(w+w') + B_5^{(3)} \cos(w+3w') + B_5^{(5)} \cos(w+5w')$$
$$+ C_5^{(1)} \cos(w-w') + C_5^{(3)} \cos(w-3w') + C_5^{(5)} \cos(w-5w') \}\,; \tag{55}$$

then we obtain

$$B_5^{(1)} = \frac{-m^2\nu_4 B_1^{(1)} - m^2\nu_2 B_3^{(1)} + B_4^{(2)}}{n^2-(n+\lambda)^2}, \quad C_5^{(1)} = \frac{-m^2\nu_4 C_1^{(1)} - m^2\nu_2 C_3^{(1)} + C_4^{(2)}}{n^2-(n-\lambda)^2}$$
$$B_5^{(3)} = \frac{-m^2\nu_2 B_3^{(3)} + B_4^{(2)} + B_4^{(4)}}{n^2-(n+3\lambda)^2}, \quad C_5^{(3)} = \frac{-m^2\nu_2 C_3^{(3)} + C_4^{(2)} + C_4^{(4)}}{n^2-(n-3\lambda)^2}$$
$$B_5^{(5)} = \frac{B_4^{(4)}}{n^2-(n+5\lambda)^2}, \quad C_5^{(5)} = \frac{C_4^{(4)}}{n^2-(n-5\lambda)^2}$$

Finally we get $B_5^{(1)}, B_5^{(3)}, B_5^{(5)}, C_5^{(1)}, C_5^{(3)}, C_5^{(5)}$ as functions of n and λ :

$$B_5^{(1)} = -\frac{116\lambda^6 - 13n\lambda^5 + 94n^2\lambda^4 + 32n^3\lambda^3 + 64n^4\lambda^2 + 80n^5\lambda + 32n^6}{6\lambda^5(\lambda^2-n^2)(\lambda^2-4n^2)^3(\lambda+2n)^2(3\lambda+2n)} \tag{56}$$

$$C_5^{(1)} = -\frac{116\lambda^6 + 13n\lambda^5 + 94n^2\lambda^4 - 32n^3\lambda^3 + 64n^4\lambda^2 - 80n^5\lambda + 32n^6}{6\lambda^5(\lambda^2 - n^2)(\lambda^2 - 4n^2)^3(\lambda - 2n)^2(3\lambda - 2n)} \tag{57}$$

$$B_5^{(3)} = \frac{13\lambda^2 + 12n\lambda + 4n^2}{35\lambda^5(\lambda + n)(\lambda^2 - 4n^2)(\lambda + 2n)^2(2\lambda + n)(3\lambda + 2n)} \tag{58}$$

$$C_5^{(3)} = \frac{13\lambda^2 - 12n\lambda + 4n^2}{35\lambda^5(\lambda - n)(\lambda^2 - 4n^2)(\lambda - 2n)^2(2\lambda - n)(3\lambda - 2n)} \tag{59}$$

$$B_5^{(5)} = -\frac{1}{480\lambda^5(\lambda + n)(\lambda + 2n)(2\lambda + n)(3\lambda + 2n)(5\lambda + 2n)} \tag{60}$$

$$C_5^{(5)} = -\frac{1}{480\lambda^5(\lambda - n)(\lambda - 2n)(2\lambda - n)(3\lambda - 2n)(5\lambda - 2n)} \tag{61}$$

3.6 6th order solution

An equation determing x_6 is

$$\ddot{x}_6 + n^2 x_6 = -m^2(\nu_6 x_0 + \nu_5 x_1 + \nu_4 x_2 + \nu_3 x_3 + \nu_2 x_4) + 2\, x_5 \cos w' \tag{62}$$

If we take in the righr hand side of (62)

$$m^2\,\nu_6 = B_5^{(1)} + C_5^{(1)} = -\frac{4(29\lambda^4 + 232n^2\lambda^2 + 144n^4)}{(\lambda^2 - n^2)(\lambda^2 - 4n^2)^5(9\lambda^2 - 4n^2)}\,, \tag{63}$$

a term of $\cos w$ does not occur in it. Now we put

$$\begin{aligned} x_6 = \eta_0\{ & B_6^{(2)} \cos(w + 2w') + B_6^{(4)} \cos(w + 4w') + B_6^{(6)} \cos(w + 6w') \\ & + C_6^{(2)} \cos(w - 2w') + C_6^{(4)} \cos(w - 4w') + C_6^{(6)} \cos(w - 6w')\}\,; \end{aligned} \tag{64}$$

then we obtain coefficients $B_6^{(2)}, B_6^{(4)}, B_6^{(6)}, C_6^{(2)}, C_6^{(4)}, C_6^{(6)}$:

$$B_6^{(2)} = \frac{-m^2\nu_4 B_2^{(2)} - m^2\nu_2 B_4^{(2)} + B_5^{(1)} + B_5^{(3)}}{n^2 - (n + 2\lambda)^2}$$

$$C_6^{(2)} = \frac{-m^2\nu_4 C_2^{(2)} - m^2\nu_2 C_4^{(2)} + C_5^{(1)} + C_5^{(3)}}{n^2 - (n - 2\lambda)^2}$$

$$B_6^{(4)} = \frac{-m^2\nu_2 B_4^{(4)} + B_5^{(3)} + B_5^{(5)}}{n^2 - (n + 4\lambda)^2}$$

$$C_6^{(4)} = \frac{-m^2\nu_2 C_4^{(4)} + C_5^{(3)} + C_5^{(5)}}{n^2 - (n - 4\lambda)^2}$$

$$B_6^{(6)} = \frac{B_5^{(5)}}{n^2 - (n + 6\lambda)^2}$$

$$C_6^{(6)} = \frac{C_5^{(5)}}{n^2 - (n - 6\lambda)^2}$$

3.7 Form of solution of Gyldén-Lindstedt equation

From the above discussion, we see that the solution of Gyldén-Lindstedt equation (1) is expressed as follows

$$x = \eta_0 \sum_{i=-\infty}^{+\infty} A_i \cos(w + iw'), \quad w = n\,t + \alpha,\ w' = \lambda\,t + \beta\,; \tag{65}$$

where

$$A_0 \;=\; 1 \tag{66}$$
$$A_1 \;=\; \varepsilon\, B_1^{(1)} + \varepsilon^3 B_3^{(1)} + \varepsilon^5 B_5^{(1)} + \cdots \tag{67}$$
$$A_{-1} = \varepsilon\, C_1^{(1)} + \varepsilon^3 C_3^{(1)} + \varepsilon^5 C_5^{(1)} + \cdots \tag{68}$$
$$A_2 \;= \varepsilon^2 B_2^{(2)} + \varepsilon^4 B_4^{(2)} + \varepsilon^6 B_6^{(2)} + \cdots \tag{69}$$
$$A_{-2} = \varepsilon^2 C_2^{(2)} + \varepsilon^4 C_4^{(2)} + \varepsilon^6 C_6^{(2)} + \cdots \tag{70}$$
$$A_3 \;= \varepsilon^3 B_3^{(3)} + \varepsilon^5 B_5^{(3)} + \cdots \tag{71}$$
$$A_{-3} = \varepsilon^3 C_3^{(3)} + \varepsilon^5 C_5^{(3)} + \cdots \tag{72}$$
$$A_4 \;= \varepsilon^4 B_4^{(4)} + \varepsilon^6 B_6^{(4)} + \cdots \tag{73}$$
$$A_{-4} = \varepsilon^4 C_4^{(4)} + \varepsilon^6 C_6^{(4)} + \cdots \tag{74}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

4 Detemination of n as a function of m and λ

We see in the precious section that the solution of Gyldén-Lindstedt equation (1) is expressed by (65), coefficients of which are functions of n and λ. But as given parameters are m and λ, so we must determine n as a function of m and λ.

From the definitions (24) and (25) we see that

$$n^2 = m^2(1-\nu) = m^2 - \varepsilon^2 m^2 \nu_2 - \varepsilon^4 m^2 \nu_4 - \varepsilon^6 m^2 \nu_6 - \cdots \tag{75}$$

4.1 Solution up to ε^2

In this case we put $n = m$ in the right hand side of (36), giving

$$\varepsilon^2 m^2\, \nu_2 = \frac{2\,\varepsilon^2}{4m^2 - \lambda^2} = -\frac{2\,\varepsilon^2}{\lambda^2 - 4m^2} \tag{76}$$

therefore we obtain as an expression up to ε^2

$$n^2 = m^2 - \frac{2\,\varepsilon^2}{4m^2 - \lambda^2} = m^2 + \frac{2\,\varepsilon^2}{\lambda^2 - 4m^2} \tag{77}$$

4.2 Solution up to ε^4

In this case, from (75)

$$n^2 = m^2 - \varepsilon^2 m^2 \nu_2 - \varepsilon^4 m^2 \nu_4 \tag{78}$$

Of the right hand side of this equation, a term of ε^2 *i.e.* $m^2\nu_2$ is given by (36), so we substitute (77) for n^2 in (36) ; and we put $n = m$ in the term of ε^4 *i.e.* $m^2\nu_4$.

First we substitute (77) in (36)

$$\begin{aligned} m^2\nu_2 &= -\frac{2}{\lambda^2 - 4\left(m^2 + \dfrac{2}{\lambda^2 - 4m^2}\right)} = -\frac{2}{\lambda^2 - 4m^2}\left\{1 + \frac{8\,\varepsilon^2}{(\lambda^2 - 4m^2)^2}\right\} \\ &= -\frac{2}{\lambda^2 - 4m^2} - \frac{16}{(\lambda^2 - 4m^2)^3}\,\varepsilon^2 \end{aligned} \tag{79}$$

Next we put $n = m$ in (47)

$$m^2\,\nu_4 = \frac{7\lambda^2 + 20m^2}{2(\lambda^2 - 4m^2)^3(\lambda^2 - m^2)} \tag{80}$$

Substitution of (79) and (80) in (78) gives

$$\begin{aligned} n^2 =& m^2 + \frac{2}{\lambda^2 - 4m^2}\,\varepsilon^2 + \frac{16}{(\lambda^2 - 4m^2)^3}\,\varepsilon^4 - \frac{7\lambda^2 + 20m^2}{2(\lambda^2 - 4m^2)^3(\lambda^2 - m^2)}\varepsilon^4 \\ =& m^2\left[1 + \frac{1}{m^2}\frac{2}{\lambda^2 - 4m^2}\,\varepsilon^2 + \frac{1}{m^2}\frac{25\lambda^2 - 52m^2}{2(\lambda^2 - 4m^2)^3(\lambda^2 - m^2)}\,\varepsilon^4\right] \end{aligned} \tag{81}$$

From this, we obtain an approximate expression up to ε^4 :

$$\begin{aligned} n =& m\left[1 + \frac{1}{2m^2}\frac{2}{\lambda^2 - 4m^2}\,\varepsilon^2 + \frac{1}{2m^2}\frac{25\lambda^2 - 52m^2}{2(\lambda^2 - 4m^2)^3(\lambda^2 - m^2)}\,\varepsilon^4 - \frac{1}{8m^4}\frac{4}{(\lambda^2 - 4m^2)^2}\,\varepsilon^4\right] \\ =& m\left[1 + \frac{1}{m^2(\lambda^2 - 4m^2)}\,\varepsilon^2 - \frac{2\lambda^4 - 35m^2\lambda^2 + 60m^4}{4m^4(\lambda^2 - 4m^2)^3(\lambda^2 - m^2)}\,\varepsilon^4\right] \end{aligned} \tag{82}$$

4.3 Solution up to ε^6

In this case we substitute (81) in $m^2\nu_2$, (77) in $m^2\nu_4$, and we put $n = m$ in $m^2\nu_6$.

Fisrt we substitute (81) in (36), giving

$$m^2\nu_2 = -\frac{2}{\lambda^2 - 4m^2} - \frac{16}{(\lambda^2 - 4m^2)^3}\,\varepsilon^2 - \frac{12(19\lambda^2 - 28m^2)}{(\lambda^2 - 4m^2)^5}\,\varepsilon^4 \tag{83}$$

Second we substitute (79) in (47), from which it results that

$$m^2\,\nu_4 = \frac{7\lambda^2 + 20m^2}{2(\lambda^2 - 4m^2)^3(\lambda^2 - m^2)} + \frac{3(37\lambda^4 + 16m^2\lambda^2 - 80m^4)}{(\lambda^2 - m^2)^2(\lambda^2 - 4m^2)^5}\,\varepsilon^2 \tag{84}$$

Third we put $n = m$ in (63), followed by

$$m^2\,\nu_6 = -\frac{4(29\lambda^4 + 232m^2\lambda^2 + 144m^4)}{(\lambda^2 - m^2)(\lambda^2 - 4m^2)^5(9\lambda^2 - 4m^2)} \tag{85}$$

Substituting (83)∼(85) in (75) and arranging it, we obtain

$$n^2 = m^2 + \frac{2}{\lambda^2 - 4m^2}\,\varepsilon^2 + \frac{25\lambda^2 - 52m^2}{2(\lambda^2 - 4m^2)^3(\lambda^2 - m^2)}\,\varepsilon^4$$

$$+\frac{1169\lambda^6 - 5164m^2\lambda^4 + 7280m^4\lambda^2 - 2880m^6}{(\lambda^2 - m^2)^2(\lambda^2 - 4m^2)^5(9\lambda^2 - 4m^2)}\,\varepsilon^6\,; \tag{86}$$

from which it follows that

$$n = m\left[1 + \frac{1}{m^2(\lambda^2 - 4m^2)}\,\varepsilon^2 - \frac{2\lambda^4 - 35m^2\lambda^2 + 60m^4}{4m^4(\lambda^2 - 4m^2)^3(\lambda^2 - m^2)}\,\varepsilon^4\right.$$

$$\left. - \frac{18\lambda^{10} - 413m^2\lambda^8 + 4705m^4\lambda^6 - 15260m^6\lambda^4 + 18480m^8\lambda^2 - 6720m^{10}}{4m^6(\lambda^2 - m^2)^2(\lambda^2 - 4m^2)^5(9\lambda^2 - 4m^2)}\,\varepsilon^6\right] \tag{87}$$

References

[1] **Gyldén, Hugo** *Über die Theorie der Bewegungen der Himmelskörper*, A.N. N°2383, 1881

[2] **Gyldén, Hugo** *Erwiederung auf die Bemerkungen des Herrn Dr. And. Lindstedt in den Astr.Nachr. N° 2462*, A.N. N°2469, 1882

[3] **Lindtstedt, And.** *Über die Integration einer für die Störungstheorie wichtigen Differentialgleichung*, A.N. N°2462, 1882

[4] **Lindtstedt, And.** *Bemerkungen zur Integration einer gewissen Differentialgleichung*, A.N. N°2465, 1882

[5] **Lindtstedt, And.** *Über die Integration einer gewissen Differentialgleichung*, A.N. N°2482, 1883

[6] **Lindtstedt, And.** *Über die allgemeine Form der Integrale des Dreikörperproblems*, A.N. N°2503, 1883

[7] **Bruns, H.** *Über eine Differentialgleichung der Störungstheorie*, A.N. N°2533, 1883

[8] **Callandreau, O.** *Sur une équation differentielle de la théorie des perturbations et remarques relatives aux N^os 2389 et 2435 des A.N.*, A.N. N°2547, 1884

[9] **Tisserand, F.** Traité de Mécanique Céleste, Tome III, pp.1 ∼ 17, Gauthier-Villars, 1894

[10] **Charlier, C.** Die Mechanik des Himmels, Bd.1, pp.22 ∼ 41, Veit, 1902

[11] **Hagihara, Yusuke** Celestial Mechanics Vol.III, pp.33 ∼ 41, Japan Society for the Promotion of Science, 1974